

A characterization of regular, intra-regular, left quasi-regular and semisimple hypersemigroups in terms of fuzzy sets

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Abstract. We prove that an hypersemigroup H is regular if and only, for any fuzzy subset f of H , we have $f \preceq f \circ 1 \circ f$ and it is intra-regular if and only if, for any fuzzy subset f of H , we have $f \preceq 1 \circ f \circ f \circ 1$. An hypersemigroup H is left (resp. right) quasi-regular if and only if, for any fuzzy subset f of S we have $f \preceq 1 \circ f \circ 1 \circ f$ (resp. $f \preceq f \circ 1 \circ f \circ 1$) and it is semisimple if and only if, for any fuzzy subset f of S we have $f \preceq 1 \circ f \circ 1 \circ f \circ 1$. The characterization of regular and intra-regular hypersemigroups in terms of fuzzy subsets are very useful for applications.

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1 Introduction

In our paper in [2] we gave, among others, a characterization of regular ordered semigroups in terms of fuzzy subsets which is very useful for applications. Using that equivalent definition of regular semigroups many known results on semigroup (without order) or on ordered semigroups can be drastically simplified. In [1], we characterized the left quasi-regular and semisimple ordered semigroups in terms of fuzzy sets. In the present paper we characterize the regular, intra-regular, left quasi-regular and semisimple hypersemigroups using fuzzy sets. According to the equivalent definition of regularity and intra-regularity given in the present paper many results on hypersemigroups can be drastically simplified. The paper has been inspired by our paper in [2], and the aim is to show the way we pass from fuzzy semigroups to fuzzy hypersemigroups. In fact, the results on semigroups or ordered semigroups can be transferred to hypersemigroups in the way indicated in the present paper.

2 Main results

An *hypergroupoid* is a nonempty set H with an hyperoperation

$$\circ : H \times H \rightarrow \mathcal{P}^*(H) \mid (a, b) \rightarrow a \circ b$$

on H and an operation

$$* : \mathcal{P}^*(H) \times \mathcal{P}^*(H) \rightarrow \mathcal{P}^*(H) \mid (A, B) \rightarrow A * B$$

on $\mathcal{P}^*(H)$ (induced by the operation of H) such that

$$A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$$

for every $A, B \in \mathcal{P}^*(H)$ ($\mathcal{P}^*(H)$ denotes the set of nonempty subsets of H).

The operation “ $*$ ” is well defined. Indeed: If $(A, B) \in \mathcal{P}^*(H) \times \mathcal{P}^*(H)$, then $A * B = \bigcup_{(a,b) \in A \times B} (a \circ b)$. For every $(a, b) \in A \times B$, we have $(a, b) \in H \times H$, then $(a \circ b) \in \mathcal{P}^*(H)$, thus we get $A * B \in \mathcal{P}^*(H)$. If $(A, B), (C, D) \in \mathcal{P}^*(H) \times \mathcal{P}^*(H)$ such that $(A, B) = (C, D)$, then

$$A * B = \bigcup_{(a,b) \in A \times B} (a \circ b) = \bigcup_{(a,b) \in C \times D} (a \circ b) = C * D.$$

As the operation “ $*$ ” depends on the hyperoperation “ \circ ”, an hypergroupoid can be also denoted by (H, \circ) (instead of $(H, \circ, *)$).

If H is an hypergroupoid then, for any $x, y \in H$, we have $x \circ y = \{x\} * \{y\}$. Indeed,

$$\{x\} * \{y\} = \bigcup_{u \in \{x\}, v \in \{y\}} u \circ v = x \circ y.$$

An hypergroupoid H is called *hypersemigroup* if

$$(x \circ y) * \{z\} = \{x\} * (y \circ z)$$

for every $x, y, z \in H$. Since $x \circ y = \{x\} * \{y\}$ for any $x, y \in H$, an hypergroupoid H is an hypersemigroup if and only if, for any $x, y, z \in H$, we have

$$(\{x\} * \{y\}) * \{z\} = \{x\} * (\{y\} * \{z\}).$$

Following Zadeh, if (H, \circ) is an hypergroupoid, we say that f is a fuzzy subset of H (or a fuzzy set in H) if f is a mapping of H into the real closed interval

$[0, 1]$ of real numbers, that is $f : H \rightarrow [0, 1]$. For an element a of H , we denote by A_a the subset of $H \times H$ defined as follows:

$$A_a := \{(y, z) \in H \times H \mid a \in y \circ z\}.$$

For two fuzzy subsets f and g of H , we denote by $f \circ g$ the fuzzy subset of H defined as follows:

$$f \circ g : H \rightarrow [0, 1] \quad a \rightarrow \begin{cases} \bigvee_{(y,z) \in A_a} \min\{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset. \end{cases}$$

Denote by $F(H)$ the set of all fuzzy subsets of H and by “ \preceq ” the order relation on $F(H)$ defined by:

$$f \preceq g \iff f(x) \leq g(x) \text{ for every } x \in H.$$

We finally show by 1 the fuzzy subset of H defined by:

$$1 : H \rightarrow [0, 1] \mid x \rightarrow 1(x) := 1.$$

Clearly, the fuzzy subset 1 is the greatest element of the ordered set $(F(H), \preceq)$ (that is, $1 \succeq f \forall f \in F(H)$).

We denote the hyperoperation on H and the multiplication between the two fuzzy subsets of H by the same symbol (no confusion is possible).

For an hypergroupoid H , we denote by f_a the fuzzy subset of H defined by:

$$f_a : H \rightarrow [0, 1] \mid x \rightarrow f_a(x) := \begin{cases} 1 & \text{if } x = a \\ 0 & \text{if } x \neq a. \end{cases}$$

The following proposition, though clear, plays an essential role in the theory of hypergroupoids.

Proposition 1. *Let (H, \circ) be an hypergroupoid, $x \in H$ and $A, B \in \mathcal{P}^*(H)$. Then we have the following:*

- (1) $x \in A * B \iff x \in a \circ b$ for some $a \in A, b \in B$.
- (2) If $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$.

Lemma 2. *If H is an hypergroupoid and A, B, C nonempty subsets of H , then $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$.*

Lemma 3. (cf. also [3; Proposition 9]) *If H is an hypersemigroup, then the set of all fuzzy subsets of H is a semigroup.*

According to this lemma, for any fuzzy subsets f, g, h of H , we write

$$(f \circ g) \circ h = f \circ (g \circ h) := f \circ g \circ h.$$

Definition 4. An hypersemigroup H is called *regular* if for every $a \in H$ there exists $x \in H$ such that

$$a \in (a \circ x) * \{a\}.$$

Equivalent Definitions:

- (1) $a \in \{a\} * H * \{a\}$ for every $a \in H$.
- (2) $A \subseteq A * H * A$ for every $A \subseteq H$.

Theorem 5. *An hypersemigroup H is regular if and only if, for any fuzzy subset f of H , we have*

$$f \preceq f \circ 1 \circ f.$$

Proof. \implies . Let f be a fuzzy subset of H and $a \in H$. Since H is regular, there exists $x \in H$ such that $a \in (a \circ x) * \{a\}$. Then, By Proposition 1(1), there exists $u \in a \circ x$ such that $a \in u \circ a$. Since $(u, a) \in A_a$, we have $A_a \neq \emptyset$ and

$$(f \circ 1 \circ f)(a) := \bigvee_{(y,z) \in A_a} \min\{(f \circ 1)(y), f(z)\} \geq \min\{(f \circ 1)(u), f(a)\}.$$

Since $(a, x) \in A_u$, we have $A_u \neq \emptyset$ and

$$(f \circ 1)(u) := \bigvee_{(y,z) \in A_u} \min\{f(y), 1(z)\} \geq \min\{f(a), 1(x)\} = f(a).$$

Thus we have

$$(f \circ 1 \circ f)(a) \geq \min\{(f \circ 1)(u), f(a)\} \geq \min\{f(a), f(a)\} = f(a).$$

\longleftarrow . Let $a \in H$. Since f_a is a fuzzy subset of H , by hypothesis, we have $1 = f_a(a) \leq (f_a \circ 1 \circ f_a)(a)$. Since $f_a \circ 1 \circ f_a$ is a fuzzy subset of H , we have $(f_a \circ 1 \circ f_a)(a) \leq 1$. Thus we have

$$(f_a \circ 1 \circ f_a)(a) = 1.$$

If $A_a = \emptyset$, then $((f_a \circ 1) \circ f_a)(a) = 0$ which is impossible. Thus we have $A_a \neq \emptyset$. Then

$$((f_a \circ 1) \circ f_a)(a) = \bigvee_{(x,y) \in A_a} \min\{(f_a \circ 1)(x), f_a(y)\}.$$

Then there exists $(x, y) \in A_a$ such that $(f_a \circ 1)(x) \neq 0$ and $f_a(y) \neq 0$. Indeed, if $(f_a \circ 1)(x) = 0$ or $f_a(y) = 0$ for every $(x, y) \in A_a$, then $\min\{(f_a \circ 1)(x), f_a(y)\} = 0$ for every $(x, y) \in A_a$, then $((f_a \circ 1) \circ f_a)(a) = 0$ which is impossible.

Since $f_a(y) \neq 0$, we have $y = a$, then $(x, a) \in A_a$. Since $(f_a \circ 1)(x) \neq 0$, we get $A_x \neq \emptyset$. Since $A_x \neq \emptyset$, we have

$$(f_a \circ 1)(x) = \bigvee_{(b,c) \in A_x} \min\{f_a(b), 1(c)\} = \bigvee_{(b,c) \in A_x} f_a(b).$$

If $b \neq a$ for every $(b, c) \in A_x$, then $f_a(b) = 0$ for every $(b, c) \in A_x$, then $(f_a \circ 1)(x) = 0$ which is impossible. Hence there exists $(b, c) \in A_x$ such that $b = a$. Then $(a, c) \in A_x$. We have $(x, a) \in A_a$ and $(a, c) \in A_x$. So we obtain $a \in x \circ a$ and $x \in a \circ c$. Then, by Lemma 2, we have

$$a \in x \circ a = \{x\} * \{a\} \subseteq (a \circ c) * \{a\}.$$

Since $c \in H$ and $a \in (a \circ c) * \{a\}$, the hypersemigroup H is regular. \square

Definition 6. An hypersemigroup H is called *intra-regular* if for every $a \in H$ there exist $x, y \in H$ such that

$$a \in (x \circ a) * (a \circ y).$$

Equivalent Definitions:

- (1) $a \in H * \{a\} * \{a\} * H$ for every $a \in H$.
- (2) $A \subseteq H * A * A * H$ for every $A \subseteq H$.

Theorem 7. An hypersemigroup H is intra-regular if and only if, for any fuzzy subset f of H , we have

$$f \preceq 1 \circ f \circ f \circ 1.$$

Proof. \implies . Let f be a fuzzy subset of H and $a \in H$. Since H is regular, there exist $x, y \in H$ such that $a \in (x \circ a) * (a \circ y)$. Then, By Proposition 1(1), there exist $u \in x \circ a$ and $v \in a \circ y$ such that $a \in u \circ v$. Since $(u, v) \in A_a$, we have

$$\begin{aligned} (1 \circ f \circ f \circ 1)(a) &= \bigvee_{(y,z) \in A_a} \min\{(1 \circ f)(y), (f \circ 1)(z)\} \\ &\geq \min\{(1 \circ f)(u), (f \circ 1)(v)\}. \end{aligned}$$

Since $(x, a) \in A_u$, we have

$$(1 \circ f)(u) = \bigvee_{(y,z) \in A_u} \min\{1(y), f(z)\} \geq \min\{1(x), f(a)\} = f(a).$$

Since $(a, y) \in A_v$, we have

$$(f \circ 1)(v) = \bigvee_{(y,z) \in A_v} \min\{f(y), 1(z)\} \geq \min\{f(a), 1(y)\} = f(a).$$

Hence we obtain

$$(1 \circ f \circ f \circ 1)(a) \geq \min\{f(a), f(a)\} = f(a),$$

so $f \preceq 1 \circ f \circ f \circ 1$.

\Leftarrow . Let $a \in H$. Since f_a is a fuzzy subset of H , by hypothesis, we have $1 = f_a(a) \leq (1 \circ f_a \circ 1 \circ f_a)(a) \leq 1$, thus $(1 \circ f_a \circ 1 \circ f_a)(a) = 1$. If $A_a = \emptyset$, then $(1 \circ f_a \circ 1 \circ f_a)(a) = 0$, impossible, thus $A_a \neq \emptyset$. Then

$$\left((1 \circ f_a) \circ (f_a \circ 1)\right)(a) = \bigvee_{(x,y) \in A_a} \min\{(1 \circ f_a)(x), (f_a \circ 1)(y)\}.$$

Then there exists $(x, y) \in A_a$ such that $(1 \circ f_a)(x) \neq 0$ and $(f_a \circ 1)(y) \neq 0$ (otherwise, $(1 \circ f_a \circ 1 \circ f_a)(a) = 0$ which is impossible). If $A_x = \emptyset$, then $(1 \circ f_a)(x) = 0$ this is no possible, thus $A_x \neq \emptyset$. Then

$$(1 \circ f_a) \circ (x) = \bigvee_{(b,c) \in A_x} \min\{1(b), f_a(c)\} = \bigvee_{(b,c) \in A_x} f_a(c).$$

If $c \neq a$ for every $(b, c) \in A_x$, then $f_a(c) = 0$ for every $(b, c) \in A_x$, then $(1 \circ f_a)(x) = 0$ which is impossible. Then there exists $(b, c) \in A_x$ such that $c = a$, so we get $(b, a) \in A_x$. If $A_y = \emptyset$, then $(f_a \circ 1)(y) = 0$, impossible, thus $A_y \neq \emptyset$, and so

$$(f_a \circ 1) \circ (y) = \bigvee_{(u,d) \in A_y} \min\{f_a(u), 1(d)\} = \bigvee_{(u,d) \in A_y} f_a(u).$$

If $u \neq a$ for every $(u, d) \in A_y$, then $f_a(u) = 0$ for every $(u, d) \in A_y$, and then $(f_a \circ 1)(y) = 0$ which is no possible. Thus there exists $(u, d) \in A_y$ such that $u = a$, then $(a, d) \in A_y$. We have $(x, y) \in A_a$, $(b, a) \in A_x$, $(a, d) \in A_y$, that is,

$$a \in x \circ y, \quad x \in b \circ a \quad \text{and} \quad y \in a \circ d.$$

Then $a \in x \circ y = \{x\} * \{y\} \subseteq (b \circ a) * (a \circ d)$, where $b, d \in H$, so the hypersemigroup H is intra-regular. \square

Definition 8. An hypersemigroup H is called *left quasi-regular* if for every $a \in H$ there exist $x, y \in H$ such that

$$a \in (x \circ a) * (y \circ a).$$

Equivalent Definitions:

- (1) $a \in H * \{a\} * H * \{a\}$ for every $a \in H$.
- (2) $A \subseteq H * A * H * A$ for every $A \subseteq H$.

Theorem 9. An hypersemigroup H is left quasi-regular if and only if, for any fuzzy subset f of H , we have

$$f \preceq 1 \circ f \circ 1 \circ f.$$

Proof. \implies . Let $a \in H$. Then $f(a) \leq (1 \circ f \circ 1 \circ f)(a)$. In fact: Since H is left quasi-regular, there exist $x, y \in H$ such that $a \in (x \circ a) * (y \circ a)$. Then there exist $u \in x \circ a$ and $v \in y \circ a$ such that $a \in u \circ v$. Since $a \in u \circ v$, we have $(u, v) \in A_a$. Since $(u, v) \in A_a$, A_a is a nonempty set and we have

$$\begin{aligned} (1 \circ f \circ 1 \circ f)(a) &:= \bigvee_{(y,z) \in A_a} \min\{(1 \circ f)(y), (1 \circ f)(z)\} \\ &\geq \min\{(1 \circ f)(u), (1 \circ f)(v)\}. \end{aligned}$$

Since $u \in x \circ a$, we have $(x, a) \in A_u$. Then A_u is a nonempty set and we have

$$(1 \circ f)(u) := \bigvee_{(s,t) \in A_u} \min\{1(s), f(t) \geq \min\{1(x), f(a)\} = f(a).$$

Since $v \in y \circ a$, we have $(y, a) \in A_v$. Then A_v is a nonempty set and we have

$$(1 \circ f)(v) := \bigvee_{(l,k) \in A_v} \min\{1(l), f(k) \geq \min\{1(y), f(a)\} = f(a).$$

Thus we get

$$(1 \circ f \circ 1 \circ f)(a) \geq \min\{f(a), f(a)\} = f(a),$$

so $f \preceq 1 \circ f \circ 1 \circ f$.

\Leftarrow . Let $a \in H$. By hypothesis, we have $1 = f_a(a) \leq (1 \circ f_a \circ 1 \circ f_a)(a) \leq 1$, so

$$(1 \circ f_a \circ 1 \circ f_a)(a) = 1.$$

If $A_a = \emptyset$, then $(1 \circ f_a \circ 1 \circ f_a)(a) = 0$ which is impossible. Thus we have $A_a \neq \emptyset$. Then

$$\left((1 \circ f_a) \circ (1 \circ f_a)\right)(a) = \bigvee_{(x,y) \in A_a} \min\{(1 \circ f_a)(x), (1 \circ f_a)(y)\}.$$

Then there exists $(x, y) \in A_a$ such that $(1 \circ f_a)(x) \neq 0$ and $(1 \circ f_a)(y) \neq 0$ (otherwise $(1 \circ f_a \circ 1 \circ f_a)(a) = 0$ which is impossible). If $A_x = \emptyset$, then $(1 \circ f_a)(x) = 0$ this is no possible, thus $A_x \neq \emptyset$. Then

$$(1 \circ f_a)(x) = \bigvee_{(b,c) \in A_x} \min\{1(b), f_a(c)\} = \bigvee_{(b,c) \in A_x} f_a(c).$$

If $c \neq a$ for every $(b, c) \in A_x$, then $f_a(c) = 0$ for every $(b, c) \in A_x$, then $(1 \circ f_a)(x) = 0$ which is impossible. Then there exists $(b, c) \in A_x$ such that $c = a$. Then we have $(b, a) \in A_x$. If $A_y = \emptyset$, then $(1 \circ f_a)(y) = 0$ which is impossible. Thus $A_y \neq \emptyset$. Then

$$(1 \circ f_a)(y) = \bigvee_{(c,d) \in A_y} \min\{1(c), f_a(d)\} = \bigvee_{(c,d) \in A_y} f_a(d).$$

If $d \neq a$ for every $(c, d) \in A_y$, then $(1 \circ f_a)(y) = 0$ which is impossible. Thus there exist $(c, d) \in A_y$ such that $d = a$. Thus we get $(c, a) \in A_y$.

We have $(x, y) \in A_a$, $(b, a) \in A_x$, $(c, a) \in A_y$, that is $a \in x \circ y$, $x \in b \circ a$, $y \in c \circ a$. Thus we have

$$a \in x \circ y = \{x\} * \{y\} \subseteq (b \circ a) * (c \circ a),$$

where $b, c \in H$, so H is left quasi-regular. \square

Definition 10. An hypersemigroup H is called *right quasi-regular* if for every $a \in H$ there exist $x, y \in H$ such that

$$a \in (a \circ x) * (a \circ y).$$

Equivalent Definitions:

- (1) $a \in \{a\} * H * \{a\} * H$ for every $a \in H$.
- (2) $A \subseteq A * H * A * H$ for every $A \subseteq H$.

The right analogue of the above theorem also holds, and we have the following theorem.

Theorem 11. *An hypersemigroup H is right quasi-regular if and only if, for any fuzzy subset f of H , we have*

$$f \preceq f \circ 1 \circ f \circ 1.$$

Definition 12. An hypersemigroup H is called *semisimple* if for every $a \in H$ there exist $x, y, z \in H$ such that

$$a \in (x \circ a) * (y \circ a) * \{z\}.$$

Equivalent Definitions:

- (1) $a \in H * \{a\} * H * \{a\} * H$ for every $a \in H$.
- (2) $A \subseteq H * A * H * A * H$ for every $A \subseteq H$.

Theorem 13. *An hypersemigroup H is semisimple if and only if, for any fuzzy subset f of H , we have*

$$f \preceq 1 \circ f \circ 1 \circ f \circ 1.$$

Proof. \implies . Let $a \in H$. Since H is semisimple, there exist $x, y \in H$ such that $a \in (x \circ a) * (y \circ a) * \{z\} = ((x \circ a) * \{y\}) * (a \circ z)$. By Proposition 1(1), there exist $u \in (x \circ a) * \{y\}$ and $v \in a \circ z$ such that $a \in u \circ v$. Since $u \in (x \circ a) * \{y\}$, By Proposition 1(1), there exists $w \in x \circ a$ such that $u \in w \circ y$. Thus we have

$$a \in u \circ v, u \in w \circ y, w \in x \circ a, v \in a \circ z.$$

Since $a \in u \circ v$, we have $(u, v) \in A_a$. Then $A_a \neq \emptyset$ and

$$\begin{aligned} (1 \circ f \circ 1 \circ f \circ 1)(a) &= \bigvee_{(c,d) \in A_a} \min\{(1 \circ f \circ 1)(c), (f \circ 1)(d)\} \\ &\geq \min\{(1 \circ f \circ 1)(u), (f \circ 1)(v)\}. \end{aligned}$$

Since $u \in w \circ y$, we have $(w, y) \in A_u$. Then $A_u \neq \emptyset$ and

$$(1 \circ f \circ 1)(u) = \bigvee_{(s,t) \in A_u} \min\{(1 \circ f)(s), 1(t)\} \geq (1 \circ f)(w).$$

Since $w \in x \circ a$, we have $(x, a) \in A_w$. Then $A_w \neq \emptyset$ and

$$(1 \circ f)(w) = \bigvee_{(\xi, \zeta) \in A_w} \min\{(1(\xi), f(\zeta))\} \geq \min\{1(x), f(a)\} = f(a).$$

Since $v \in a \circ z$, we have $(a, z) \in A_v$. Then $A_v \neq \emptyset$ and

$$(f \circ 1)(v) = \bigvee_{(k, h) \in A_v} \min\{f(k), 1(h)\} = \bigvee_{(k, h) \in A_v} f(k) \geq f(a).$$

Thus we have

$$(1 \circ f \circ 1 \circ f \circ 1)(a) \geq \min\{f(a), f(a)\} = f(a),$$

so $f \preceq 1 \circ f \circ 1 \circ f \circ 1$.

\Leftarrow . Let $a \in H$. By hypothesis, we have $(1 \circ f_a \circ 1 \circ f_a \circ 1)(a) = 1$. Then $A_a \neq \emptyset$ and

$$(1 \circ f_a \circ 1 \circ f_a \circ 1)(a) = \bigvee_{(x, y) \in A_a} \min\{(1 \circ f_a \circ 1)(x), (f_a \circ 1)(y)\}.$$

Then there exists $(x, y) \in A_a$ such that $(1 \circ f_a \circ 1)(x) \neq 0$ and $(f_a \circ 1)(y) \neq 0$.

If $A_y = \emptyset$, then $(f_a \circ 1)(y) = 0$ which is impossible. Thus $A_y \neq \emptyset$ and

$$(f_a \circ 1)(y) = \bigvee_{(b, c) \in A_y} \min\{f_a(b), 1(c)\} = \bigvee_{(b, c) \in A_y} f_a(b).$$

If $b \neq a$ for every $(b, c) \in A_y$, then $f_a(b) = 0$ for every $(b, c) \in A_y$, then $(f_a \circ 1)(y) = 0$ which is impossible. Thus there exists $(b, c) \in A_y$ such that $b = a$, then $(a, c) \in A_y$. If $A_x = \emptyset$, then $(1 \circ f_a \circ 1)(x) = 0$, impossible. Thus $A_x \neq \emptyset$ and

$$(1 \circ f_a \circ 1)(x) = \bigvee_{(\rho, \lambda) \in A_x} \min\{1(\rho), (f_a \circ 1)(\lambda)\} = \bigvee_{(\rho, \lambda) \in A_x} (f_a \circ 1)(\lambda).$$

If $(f_a \circ 1)(\lambda) = 0$ for every $(\rho, \lambda) \in A_x$, then $(1 \circ f_a \circ 1)(x) = 0$, impossible. Then there exists $(\rho, \lambda) \in A_x$ such that $(f_a \circ 1)(\lambda) \neq 0$. If $A_\lambda = \emptyset$, then $(f_a \circ 1)(\lambda) = 0$, impossible. Thus $A_\lambda \neq \emptyset$ and

$$(f_a \circ 1)(\lambda) = \bigvee_{(k, h) \in A_\lambda} \min\{f_a(k), 1(h)\} = \bigvee_{(k, h) \in A_\lambda} f_a(k).$$

If $a \neq k$ for every $(k, h) \in A_\lambda$, then $(f_a \circ 1)(\lambda) = 0$, impossible. Thus there exists $(k, h) \in A_\lambda$ such that $a = k$, then $(a, h) \in A_\lambda$. We have

$$a \in x \circ y, y \in a \circ c, x \in \rho \circ \lambda, \lambda \in a \circ h.$$

Then we have

$$\begin{aligned} a \in \{x\} * \{y\} &\subseteq \{\rho\} * \{\lambda\} * \{a\} * \{c\} \subseteq \{\rho\} * \{a\} * \{h\} * \{a\} * \{c\} \\ &= (\rho \circ a) * (h \circ a) * \{c\}, \end{aligned}$$

where $\rho, h, c \in H$, so H is semisimple. \square

Note. The characterization of regular and intra-regular hypersemigroups in terms of fuzzy sets given in this paper are very useful for further investigation. Exactly as in semigroups, using these definitions, many proofs on hypersemigroups can be drastically simplified. Let us just give an example to clarify what we say, further interesting information concerning this structure will be given in a forthcoming paper. We begin with the definition of fuzzy right and fuzzy left ideals of hypersemigroups. If H is an hypersemigroup, a fuzzy subset of H is called a *fuzzy right ideal* of H if $f(x \circ y) \geq f(x)$ for every $x, y \in H$, in the sense that if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(x)$. A fuzzy subset of H is called a *fuzzy left ideal* of H if $f(x \circ y) \geq f(y)$ for every $x, y \in H$, that is, if $x, y \in H$ and $u \in x \circ y$, then $f(u) \geq f(y)$. A fuzzy subset f of H is a fuzzy right (resp. fuzzy left) ideal of H if and only if $f \circ 1 \preceq f$ (resp. $1 \circ f \preceq f$) [3]. Using the definition of regular hypersemigroups given in the present paper, one can immediately see that if H is a regular hypersemigroup then, for every fuzzy right ideal f and every fuzzy left ideal g of H , we have $f \wedge g = f \circ g$. In fact,

$$\begin{aligned} f \wedge g &\preceq (f \wedge g) \circ 1 \circ (f \wedge g) \preceq (f \circ 1) \circ g \preceq f \circ g \\ &\preceq (f \circ 1) \wedge (1 \circ g) \preceq f \circ g, \end{aligned}$$

thus $f \wedge g = f \circ g$.

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